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The reduction of one-dimensional eigenvalue problems to the solution of simultaneous algebraic equations with one nonlinearity

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SUMMARY

A method is developed to obtain numerically eigenvalues associated with ordinary differential equations and variational formulations. The numerical operation required is the solution of a set of linear algebraic equations, with one non-linearity, i.e., the transcendental dependence on the eigenvalue. This operation is the same for all problems resulting from equations of the same order, and therefore can easily be programmed; the particulars of a specific problem are then just input for the program.

1. Introduction

Let an eigenvalue problem be given by

$$G_n(y, \lambda) = 0\,, \tag{1}$$

where G_n is a linear differential operator of order n, y an eigenfunction, and λ the corresponding eigenvalue. Eq. (1) is associated with a proper set of homogeneous boundary conditions. This eigenvalue problem is assumed to have an equivalent variational formulation

$$\lambda = \text{Min}\left[\int F(y)d\xi \,/\, \int H(y)d\xi\right], \tag{2}$$

where F and H are differential operators and the integration is over the proper region.

In general G_n will generate a linear ordinary differential equation of the nth order *with variable coefficients*. Thus, in general the solution of the eigenvalue problem will not be known.

Because the motivation for this investigation stems from these solutions being "unknown", this point must be further elaborated, even at the risk of being too obvious.

What is usually meant by a solution being known is that it has a "name" (associated sometimes with the "name" of the equation) and that its numerical values, including its zeros are directly available, either by computation or from tables. Examples of such known solutions are *sin*, *cos*, *Bessel* functions, *Legendre* polynomials, and some other particular cases of the second order linear equation associated with the Sturm–Liouville problem (*e.g.*, Ref. [1], [2]). All these known solutions are obtained (at least one of each pair) by power series expansion known as the method of Frobenius Ref. [2]. Once one solution is known, the other one is directly obtainable also (*e.g.*, Ref. [2]).

While this method is applicable, in principle, to a wide variety of equations, including higher order linear equations, Ref. [2], or non-linear ones, Ref. [3], there are cases where it is quite impractical, even when computers are used. The main obstacle here is that the coefficients of the series contain the eigenvalues, and when boundary conditions are to be satisfied at large values of the independent variable this may become quite cumbersome. Ref. [3] is an example where this difficulty arises, but is overcome, and Ref. [4] is an example where auxiliary methods must be employed.

Since practical engineering situations require the knowledge of such solutions, effort is being made to find bounds to the eigenvalues, or approximate values. Some indication of this endeavor is found in Refs. [5]–[32]. Indeed, the work presented in this paper is directed toward another method to obtain bounds and approximations to eigenvalues.

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While the first eigenvalues in many problems can be bounded or approximated by many methods, the same does not apply to higher eigenvalues. Only few methods apply then, with the difficulties mounting rapidly, the higher the eigenvalue. Methods which apply to higher eigenvalues are usually restricted to particular cases of Eqs. (1) and (2), *e.g.*, Refs. [11], [13], [14], [17], [21], [23], [24] and [26].

The contribution of this paper is that it presents a method which:

(a) Yields bounds to the eigenvalues (the first and the higher ones) for a large class of operators (eqs. (1)).

(b) Yields approximations, which can be systematically improved, to the eigenvalues (the first and higher ones) for the most general case of operators (eqs. (1) and (2)).

(c) Yields approximations, which can be systematically improved, to the eigenfunctions and to the zeros (the first and the higher ones) for the most general case of operators (eqs. (1) and (2)).

2. Analysis

Let an auxiliary operator be defined

$$K_n(\eta, \mu) = 0\,, \tag{3}$$

as that obtained by changing each coefficient in G_n into some constant (to be chosen later). Obviously K_n generates a linear ordinary differential equation of the nth order *with constant coefficients*.

This work is based on the assumption that the general solution of Eq. (3) is analytically known.

Let an approximation to y (the exact solution of the eigenvalue problem, i.e., Eq. (1)), be defined

$$\eta \sim y \tag{4}$$

i.e., η is an approximation to y, if and only if:

(i) The region over which y is defined can be broken into a finite number of subregions.

(ii) η satisfies eq. (3) over each such subregion; also η satisfies the same boundary conditions as y. (It is implied that η and its n first derivatives are continuous over the open subregions.)

(iii) η and its first $(n-1)$ derivatives are continuous everywhere, including over the boundaries of the subregions.

Because the general solution of eq. (3) is analytically known, the form of η over every subregion is the sum of n known functions with n coefficients (different for each subregion). Let the number of subregions be m. Then the total number of coefficients is $(m \times n)$. It is noted that because the boundary conditions and eq. (3) are homogeneous, one of the coefficients can be arbitrarily chosen, thus only $(m \times n) - 1$ of them need be computed. However, μ must also be obtained. Hence, to completely determine the approximation, μ and $(m \times n) - 1$ coefficients are needed, i.e., $(m \times n)$ numbers.

The definition of the approximation supplies the $(m \times n)$ necessary equations: n as boundary conditions and $n \times (m-1)$ as continuity conditions. A general method to solve this set of equations is considered later.

There are, of course, many approximations to y, depending on the number of the subregions and on their distribution. It is, therefore, helpful to define a hierarchy of approximations:

Let μ_j and μ_i be the eigenvalues corresponding to the approximations η_j and η_i.

η_j is defined as a *better approximation than* η_i if and only if:

$$|\lambda - \mu_j| < |\lambda - \mu_i|\,. \tag{5}$$

It is noted that the additional information

$$\lambda > \mu_i \text{ or } \lambda < \mu_i \tag{6}$$

may be very important in many practical cases. Considerations are, therefore, given first to cases where $\lambda > \mu_i$, i.e., where μ_i is a lower bound* to the exact value.

Let the operator $K_n(\eta, \mu)=0$ be associated with an equivalent variational formulation (similarly to the relations between eq. (1) and eq. (2)):

$$\mu = \text{Min}\left[\int L(\eta)d\xi \,/\, \int M(\eta)d\xi\right]. \tag{7}$$

Let the operators in eq. (2) have the form**

$$F(y) = \sum_{k=0}^{n/2} f_k \left(\frac{d^k y}{d\xi^k}\right)^2, \tag{8}$$

$$H(y) = \sum_{k=0}^{k<n/2} h_k \left(\frac{d^k y}{d\xi^k}\right)^2. \tag{9}$$

The form of $L(\eta)$ and $M(\eta)$, eq. (7), will then be:

$$L(\eta) = \sum_{k=0}^{n/2} a_k \left(\frac{d^k \eta}{d\xi^k}\right)^2, \tag{10}$$

$$M(\eta) = \sum_{k=0}^{k<n/2} b_k \left(\frac{d^k \eta}{d\xi^k}\right)^2, \tag{11}$$

with a_k and b_k constants, being different over each subregion

$$\xi_{L-1} \leqq \xi \leqq \xi_L .$$

A systematic generation of the coefficients a_k and b_k over the proper subregion is now suggested, so as to guarantee $\mu \leqq \lambda$:

(i) When it is *known* that everywhere over the Lth subregion

$$\left[\frac{d}{d\xi}\left(\frac{d^k y}{d\xi^k}\right)^2\right] \times \left[\frac{d}{d\xi} f_k\right] > 0, \tag{12}$$

choose

$$a_k = \frac{1}{\xi_L - \xi_{L-1}} \int_{\xi_{L-1}}^{\xi_L} f_k d\xi, \tag{13}$$

Ref. [28].

(ii) Otherwise, choose

$$a_k = \text{Min}\left[f_k\right]_{\xi_{L-1} \leqq \xi \leqq \xi_L}. \tag{14}$$

Ref. [33].

(iii) When it is *known* that

$$\left[\frac{d}{d\xi}\left(\frac{d^k y}{d\xi^k}\right)^2\right] \times \left[\frac{d}{d\xi} h_k\right] < 0 \tag{15}$$

everywhere over the Lth subregion, choose

$$b_k = \frac{1}{\xi_L - \xi_{L-1}} \int_{\xi_{L-1}}^{\xi_L} h_k d\xi. \tag{16}$$

(iv) Otherwise, choose

$$b_k = \text{Max}\left[h_k\right]_{\xi_{L-1} \leqq \xi \leqq \xi_L}. \tag{17}$$

Ref. [33].

In the language of the references cited, the operations indicated in (i) and (iii) constitute "shifts" and therefore lead to lower bounds (refs. [27] and [28]); while those of (ii) and (iv) are

* Upper bounds are assumed to be directly available by the Rayleigh–Ritz method. Still, a corresponding argument can be applied to upper bounds too.

** A more general form is still admissible, using a slightly modified argumentation.

bounds to weight functions, and also lead to bounds (Ref. [33]). Furthermore, a direct result* of the analysis presented in those references is that further subdivision of the subregions yield better (and, therefore, higher) lower bounds to the eigenvalues. Hence, the process of repeated subdivisions creates monotonically increasing series of the approximations to the eigenvalues, which is bounded from above by the exact values, and, therefore, it must converge. When this scheme can be completed, i.e., the coefficients and the eigenvalue can be obtained for each approximation, this satisfies properties (a) and (b) in the Introduction: The approximations are monotonically improved, and the eigenvalues are bounded from below.

This method is always applicable (note that to apply (ii), eq. (14) it is not necessary that eq. (12) does not hold; the same is noted about eqs. (17) and (15)). Still in certain cases a modification may be helpful. In some cases, a better approximation to the exact value may be more important than the information resulting from its being a bound, i.e., one is ready to give up this information in order to have a faster convergence. Also, there may be cases where eqs. (14) and (17) yield the value zero. When this zero is the coefficient of the highest derivative the case requires a special treatment in the numerical solution, which one may prefer to avoid.

It is stated that when eqs. (13) and (16) are always used, the approximations also form a series of "monotonically better ones", in the sense of Eq. (5).

3. Examples

In order to illustrate the method we shall address ourselves to the following two examples of the Sturm–Liouville boundary value problem:

Case 1: Zeroth order Bessel function

$$\xi \frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} + \lambda^2 \xi y = 0\,, \tag{18}$$

with the boundary conditions

$$y'(0) = 0 \text{ and } y(1) = 0\,.$$

Case 2: Temperature distribution in laminar flow within an annulus

$$\xi \frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} + \frac{2\lambda^2 \xi \left(1-\xi^2 - \dfrac{1-\beta^2}{\ln \beta} \ln \xi\right)}{R_0 \left(1+\beta^2 + \dfrac{1-\beta^2}{\ln \beta}\right)} y = 0\,, \tag{19}$$

where

$$\beta = \frac{R_i}{R_0} = \frac{1}{6}, \qquad R_0 = 6\,,$$

with boundary conditions

$$y(\beta) = 0 \text{ and } y'(1) = 0\,.$$

Both these problems were solved by a general computer program for Sturm–Liouville problems. The coefficients for each subregion were computed by eqs. (13) and (16). For this purpose the indefinite integrals of f_k and h_k were supplied as input to the program (more complicated functions can be treated by numerical integration). The resulting constants for each subregion yield a functional form for η which is either in terms of trigonometric or hyperbolic functions. The requirement that η and its first derivative be continuous at the

* An outline of a proof is that by merging two subregions into one, and performing the operation described in Refs. [27], [28] or [33], as the case may be, a lower bound (corresponding to the merged subregion) to a lower bound (corresponding to the two original subregions) is obtained. Thus the two subregions yield a higher and therefore better lower bound.

boundary between subregions, along with the boundary conditions (which are input to the program) yield the desired system of equations. This system is linear except for the approximating eigenvalue, μ. For any guess of μ the system having a band width of two can be solved directly with no inversion, and the last equation used as a check on the accuracy of the guess of μ. Thus the problem reduces to searching for the root of a transcendental equation, which then yields the value of μ.

When the program finds an interval where an eigenvalue (first or higher) exists then a search by the Regula-falsi method is made, using only a total of two subregions. The determined value of μ is then refined by successively doubling the number of subregions and using a quadratic search until the desired level of accuracy is attained.

Table 1 shows the computer output for the first eigenvalues of the zeroth order Bessel function. It is interesting to note that even the first approximation of 2.45 for two subregions is fairly close to the actual eigenvalue.

TABLE 1

Sequence of convergence of first eigenvalue of the zeroth order Bessel function.

Case 1: Zeroth order Bessel function

EIGENVALUE NUMBER 1

Number of subregions	*Eigenvalue*	*Relative error*	*Function evaluations*
2	2.44957985016	not known	8
4	2.43396768647	6.41429E-03	4
8	2.41618816789	7.35850E-03	7
16	2.40867108730	3.12084E-03	7
32	2.40603829591	1.09424E-03	6
64	2.40519164351	3.52010E-04	5
128	2.40493280915	1.07626E-04	5
256	2.40485630312	3.18131E-05	5
512	2.40483422715	9.17983E-06	5
1024	2.4048279083	2.60157E-06	5
2048	2.40482622242	7.27042E-07	4
4096	2.40482573924	2.00922E-07	4
8192	2.40482560692	5.50210E-08	4
16384	2.40482557096	1.49528E-08	4
32768	2.40482556125	4.03749E-09	4

TABLE 2

Summary of results for first ten eigenvalues for temperature distribution for laminar flow in an annulus.

Case 2: Temperature field in laminar flow through an annulus

Number	*Eigenvalue*	*Number of subregions*	*Relative error*
1	1.12495397498	8192	6.00478E-09
2	4.58268509350	8192	3.66127E-09
3	7.83072653894	8192	6.00827E-09
4	11.04763008738	8192	8.24317E-09
5	14.25302968271	16384	2.56378E-09
6	17.45276712492	16384	3.01836E-09
7	20.64925580094	16384	3.43319E-09
8	23.84369012742	16384	3.81547E-09
9	27.03673453524	16384	4.17067E-09
10	30.22879043503	16384	4.50300E-09

Table 2 summarizes results for the first ten eigenvalues for the temperature distribution in laminar flow in an annulus. The results are in agreement with those obtained by conventional methods [34]. As seen, the number of subregions required for a given level of accuracy is about the same for each eigenvalue.

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